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ON THE CONSTRUCTION OF STABLE BRIDGES*

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A method of constructing stable bridges is described, and a class of games for which this method makes it possible to construct such bridge in explicit form is indicated.

1. Let us consider a controlled process whose equations of motion are of the form

$$z^{\cdot} = f(z, u, v), \quad z \in R^n, \quad u \in U \subset R^n, \quad v \in V \subset R^n \tag{1.1}$$

A closed set Z is specified in R^n. The maximal u-stable bridge /1/ leading to the target Z at the specified time is to be constructed.

Polynomial mapping T_σ /2/ may be used for constructing the bridge. Let the set $X \subset R^n$ and the number $\sigma \geqslant 0$ be specified. Then $T_\sigma(X)$ is a set of points $z \in R^n$ for each of which it is possible to indicate on any measurable control $v(t) \in V$ a measurable control $u(t) \in U$ such that $z(\sigma) \in X$, where $z(\sigma)$ is the solution of system (1.1) with the initial condition $z(0) = z$.

By imposing on the right-hand side of system (1.1) and on the sets U and V certain conditions we impart on the mapping T_σ the following properties: 1) if set X is closed, then $T_\sigma(X)$ is also closed; 2) if $X \subset X_1$, then $T_\sigma(X) \subset T_\sigma(X_1)$; 3) $T_0(X) = X$, and 4) $T_{\sigma_1}(T_{\sigma_2}(X)) \subset T_{\sigma_1+\sigma_2}(X)$.

A collection of closed sets (which we denote by $W(t)$) that satisfy the inclusion $T_\sigma(W(t-\sigma)) \supset W(t)$ and the equality $W(0) = Z$ was constructed in /2/ using mapping T_σ. This collection of sets satisfies among other properties, the following maximality condition: if $z \in W(t)$, there exists finite set of positive numbers $\sigma_1, \ldots, \sigma_k$ whose sum equals t, and $z \in T_{\sigma_1} \cdot (\ldots T_{\sigma_k}(Z) \ldots)$.

Let us show another scheme of constructing a maximal u-stable bridge $W(t)$. For integral $k \geqslant 1$ we set $W^k(0) = Z$, and for $t > 0$ we determine $W^k(t)$ by the recurrent formula

$$W^1(t) = T_t(Z), \ldots, W^{k+1}(t) = \bigcap_{0 \leqslant \tau \leqslant t} T_\tau(W^k(t-\tau)) \tag{1.2}$$

For every $k \geqslant 1$ and $t \geqslant 0$ the constructed sets are closed.

Lemma 1.1. $W^{k+1}(t) \subset W^k(t)$ when $k > 1$ and $t \geqslant 0$.

Proof. When $k = 1$ the required inclusion follows from (1.2) and property 4 of mapping T_σ.

Let for some $k > 1$ the required inclusion is satisfied for all $t \geqslant 0$. Then using property 2 of mapping T_σ, we obtain

$$W^{k+2}(t) = \bigcap_{0 \leqslant \tau \leqslant t} T_\tau(W^{k+1}(t-\tau)) \subset \bigcap_{0 \leqslant \tau \leqslant t} T_\tau(W^k(t-\tau)) = W^{k+1}(t)$$

Lemma 1.2. For any $t > 0$ and any set $\sigma_1, \ldots, \sigma_k$, consisting of k positive numbers whose sum is equal t the inclusion

$$W^k(t) \subset T_{\sigma_1}(\ldots T_{\sigma_k}(Z) \ldots)$$

is satisfied.

Proof. As implied by (1.2) the required inclusion is satisfied when $k = 1$. Let the inclusion which is being proved be satisfied for some $k \geqslant 1$ and all $t > 0$. Then, as implied by (1.2) and property 2 of mapping T_σ the inclusion

$$W^{k+1}(t) \subset T_{\sigma_1}(W^k(t-\sigma_1)) \subset T_{\sigma_1}(T_{\sigma_2} \cdot (\ldots T_{\sigma_{k+1}}(Z) \ldots))$$

is satisfied for any set $\sigma_1, \ldots, \sigma_{k+1}$ consisting of $k+1$ positive numbers whose sum is equal t.

Theorem. $W(t) = \bigcap_{k \geqslant 0} W^k(t)$.

Proof. When $t = 0$, the left- and right-hand sides of this equality contain the set Z.

Consider the case of $t > 0$. By induction with respect to k we shall prove that $W(t) \subset W^k(t)$ for all $k \geqslant 1$. When $k = 1$ we have

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$$W(t) \subset T_t(W(0)) = T_t(Z) = W^1(t)$$

Let the inclusion that is being proved be satisfied for some $k \geqslant 1$ for all $t > 0$. Then for $0 \leqslant \tau \leqslant t$

$$W(t) \subset T_\tau(W(t-\tau)) \subset T_\tau(W^k(t-\tau))$$

from which and (1.2) follows the inclusion $W(t) \subset W^{k+1}(t)$.

Let us now show that $W(t) \supset \bigcap W^k(t)$. Let the point $z \overline{\in} W(t)$. Then, as implied by the maximality condition of bridge $W(t)$, there exists a finite set of positive numbers $\sigma_1, \ldots, \sigma_i$ whose sum is equal t, and $z \overline{\in} T_{\sigma_1}(\ldots T_{\sigma_i}(Z)\ldots)$.

This and Lemma 1.2 show that $z \overline{\in} W^i(t)$. Hence $z \overline{\in} \bigcap W^k(t)$ and, consequently, the required inclusion is proved.

Corollary 1. Let there exist numbers $t_0 > 0$ and $k \geqslant 1$ such that $W^{k+1}(t) = W^k(t)$ for all $0 \leqslant t \leqslant t_0$. Then $W(t) = W^k(t)$ for $0 \leqslant t \leqslant t_0$.

Proof. It follows from the condition that $W^{k+1}(t) = W^k(t)$ and (1.2) that $W^k(t) = W^i(t)$ for all $i \geqslant k$ and $0 \leqslant t \leqslant t_0$. The required equality follows from this and the previous theorem.

Corollary 2. Let the conditions of the preceding corollary be satisfied, and let there exist a sequence $t_i \to t_0, t_i > t_0$ such that the sets $W^k(t_i)$ are empty. Then $W(t) = W^{k+1}(t)$ for all $t \geqslant 0$.

Proof. Let us show that $W^{k+1}(t) = W^{k+2}(t)$ for all $t \geqslant 0$. For this it is sufficient to show that the set $W^{k+1}(t)$ is empty for any $t > t_0$.

Let us take t_i such that $\tau = t - t_i > 0$. It follows then from (1.2) that $W^{k+1}(t) \subset T_\tau(W^k(t_i))$. The set in the right-hand side of this inclusion is empty.

2. Let us consider the linear game

$$z^\cdot = Cz - u + v,\ z \in R^n,\ u \in U \subset R^n,\ v \in V \subset R^n \tag{2.1}$$

where C is a constant matrix, and U and V convex compacts.

We assume that an m-dimensional Euclidean space R^m and the linear mapping $\pi : R^n \to R^m$ are specified. A closed set E is specified in R^m. The terminal set Z is specified as follows:

$$Z = \{z \in R^n : \pi z \in E\} \tag{2.2}$$

We introduce the notation

$$\pi_1(t) = \pi e^{tC}, \quad J_1(t_1, t_2) = \int_{t_1}^{t_2} \pi_1(t)\, U dt, \quad J_2(t_1, t_2) = \int_{t_1}^{t_2} \pi_1(t)\, V dt \tag{2.3}$$

Then using the definition of the geometric remainder $\overset{*}{-}$ of two sets /3/, it is possible to show that for any $0 \leqslant \tau \leqslant t$ the set $T_\tau(T_{t-\tau}(Z))$ is the aggregate $z \in R^n$ of the form

$$\pi_1(t) z \in (((E + J_1(0, t-\tau)) \overset{*}{-} J_2(0, t-\tau)) + J_1(t-\tau, t)) \overset{*}{-} J_2(t-\tau, t) \tag{2.4}$$

Assumption 1. There exists in R^m the basis $x_1, \ldots, x_m$ and continuous functions $a_i(t) \leqslant A_i(t),\ i = 1, \ldots, m$ such that for any $t \geqslant 0$

$$\pi_1(t)\, U = \{x \in R^m : a_i(t) \leqslant (x_i, x) \leqslant A_i(t),\ i = 1, \ldots, m\} \tag{2.5}$$

where (x_i, x) is a scalar product.

Note some of the properties of polyhedrons of form (2.5).

Let numbers $p_i \leqslant P_i,\ i = 1, \ldots, m$. be specified. We set

$$P = \{x \in R^m : p_i \leqslant (x_i, x) \leqslant P_i,\ i = 1, \ldots, m\} \tag{2.6}$$

Lemma 2.1. Let B be a compact in R^m, and $b_i = \min(x_i, x)$ and $B_i = \max(x_i, x)$, where min and max are taken with respect to $x \in B$. Then

$$P \overset{*}{-} B = \{x \in R^m : p_i - b_i \leqslant (x_i, x) \leqslant P_i - B_i,\ i = 1, \ldots, m\} \tag{2.7}$$

Proof. Let x belong to the set appearing in the right-hand side of (2.7). Let us take any vector $y \in B$ and show that $x + y \in P$. We have $p_i - b_i \leqslant (x_i, x) \leqslant P_i - B_i$ and $b_i \leqslant (x_i, y) \leqslant B_i$. Adding these inequalities and taking into account the form of set (2.6) we obtain the required inclusion.

Let $x \in P \overset{*}{-} B$. Then $x + y \in P$ for any $y \in B$. Hence $p_i \leqslant (x_i, x) + (x_i, y) \leqslant P_i$ for any $y \in B$. Thus, taking into account the definition of numbers b_i and B_i, we conclude that x belongs to the set appearing in the right-hand side of (2.7).

Lemma (2.2). Let the numbers $b_i \leqslant B_i, i = 1, \ldots, m$ be given. We set

$$B = \{x \in R^m : b_i \leqslant (x_i, x) \leqslant B_i,\ i = 1, \ldots, m\}$$

Then

$$P + B = \{x \in R^m : p_i + b_i \leqslant (x_i, x) \leqslant P_i + B_i,\ i = 1, \ldots, m\} \tag{2.8}$$

Proof. Let x belong to set M which appears at the right-hand side of (2.8). If we can find a vector $y \in B$ such that $x - y \in P$, the inclusion $x \in P + B$ will be proved.

The inequalities that are satisfied by the numbers (x_i, x) imply that

$$(x_i, x) = (P_i + B_i + p_i + b_i + \lambda_i (P_i + B_i - p_i - b_i)) / 2,\ |\lambda_i| \leqslant 1 \tag{2.9}$$

Since $x_1, \ldots, x_m$ is the basis in R^m, there exists a vector $y \in R^m$ such that

$$(x_i, y) = (B_i + b_i + \lambda_i (B_i - b_i)) / 2$$

From this and (2.9) follows that

$$(x_i, x - y) = (P_i + p_i + \lambda_i (P_i - p_i)) / 2$$

Taking into account that $|\lambda_i| \leqslant 1$, we obtain $y \in B$ and $x - y \in P$.

The inclusion $P + B \subset M$ directly follows from the form of sets P and B.

Using Lemma 2.2. and Assumption 1 it is possible to show that the set $J_1(t_1, t_2)$, (2.3) is of the form

$$J_1(t_1, t_2) = \left\{x \in R^m : \int_{t_1}^{t_2} a_i(t)\,dt \leqslant (x_i, x) \leqslant \int_{t_1}^{t_2} A_i(t)\,dt,\ i = 1, \ldots, m\right\} \tag{2.10}$$

Assumption 2. There exist numbers $\varepsilon_i \leqslant \beta_i$ $(i = 1, \ldots, m)$ such that the set E in equality (2.2) is of the form

$$E = \{x \in R^m : \varepsilon_i \leqslant (x_i, x) \leqslant \beta_i,\ i = 1, \ldots, m\}$$

We introduce the notation /

$$b_i(t) = \min_x (x_i, x),\ B_i(t) = \max_x (x_i, x),\ x \in \pi_1(t) V,\quad \mu_i(t) = \int_0^t (a_i(\tau) - b_i(\tau))\,d\tau,\quad \nu_i(t) = \int_0^t (A_i(\tau) - B_i(\tau))\,d\tau$$

set $\tau = 0$ in (2.4) and, using Lemmas 2.1 and 2.2 and equality (2.10), obtain

$$W^1(t) = \{z \in R^n : \varepsilon_i + \mu_i(t) \leqslant (x_i, \pi_1(t) z) \leqslant \beta_i + \nu_i(t),\quad i = 1, \ldots, m\} \tag{2.11}$$

Let

$$t_0 = \sup\{t \geqslant 0 : \varepsilon_i + \mu_i(\tau) \leqslant \beta_i + \nu_i(\tau),\quad 0 \leqslant \tau \leqslant t,\quad i = 1, \ldots, m\} \tag{2.12}$$

Then for all $0 \leqslant t \leqslant t_0$ set (2.11) is nonempty. It follows from (2.4), Lemmas 2.1 and 2.2 that $T_\tau(T_{t-\tau}(Z)) = T_t(Z)$ for $0 \leqslant \tau \leqslant t \leqslant t_0$. Hence $W^2(t) = W^1(t)$ for $0 \leqslant t \leqslant t_0$.

Let $t_0 = +\infty$, then from Corollary 1 we obtain $W(t) = W^1(t)$ for all $t \geqslant 0$.

If $t_0 < +\infty$, then from the definition of the number t_0 in (2.12) and from equality (2.11) follows that there exists a sequence of numbers $t_i \to t_0$, $t_i > t_0$ such that the sets $W^1(t_i)$ are empty. By virtue of Corollary 2 $W(t) = W^2(t)$. In other words, $W(t) = T_t(Z)$ for $0 \leqslant t \leqslant t_0$ and the set $W(t)$ is empty for $t > t_0$.

3. The result obtained in Sect.2 can be used for solving the game problem with fixed time t_1 and final payoff defined by

$$g(z(t_1)) = \max_{1 \leqslant i \leqslant m} |(x_i, \pi z(t_1))| \tag{3.1}$$

Having selected control u, the first player strives to minimize the quantity (3.1), while the second tries to maximize it.

For the determination of the value $G(z)$ of such game (z is the initial position) we follow /2/. For each $\beta \geqslant 0$ we set

$$E(\beta) = \{x \in R^m : -\beta \leqslant (x_i, x) \leqslant \beta,\ i = 1, \ldots, m\}$$

Then, as implied by (3.1), the set $Z(\beta)$ of those points $z \in R^n$ at which $g(z) \leqslant \beta$ is of the form

$$Z(\beta) = \{z \in R^n : \pi z \in E(\beta)\} \tag{3.2}$$

We denote by $W(t_1, \beta)$ the stable bridge which leads to target (3.2). In this notation the value of the game is defined by

$$G(z) = \min\{\beta \geqslant 0 : z \in W(t_1, \beta)\} \tag{3.3}$$

If we set in (2.11) and (2.12) $\varepsilon_i = -\beta, \beta_i = \beta$, the number t_0 in (2.12) depends on β, i.e. $t_0 = t(\beta)$. From (3.3), (2.11), and (2.12) we obtain that $G(z)$ is equal to the smallest of numbers $\beta \geqslant 0$ that satisfy the following two inequalities:

$$t_1 \leqslant t(\beta)$$
$$\max_{1 \leqslant i \leqslant m} (|(x_i, \pi_1(t_1) z) - (\nu_i(t_1) + \mu_i(t_1)) / 2| + (\mu_i(t_1) - \nu_i(t_1)) / 2) \leqslant \beta$$

of which the first shows that the set $W(t_1, \beta)$ is nonempty, and the second is equivalent to the inclusion $z \in W(t_1, \beta)$.

As an example, let us consider the game

$$z_1^{\cdot} = z_3 + v_1, \quad z_2^{\cdot} = z_4 + v_2, \quad v_1^2 + v_2^2 \leqslant 1$$
$$z_3^{\cdot} = u_1, \quad z_4^{\cdot} = u_2, \quad |u_1| \leqslant 1, \quad |u_2| \leqslant 1$$

Let the final payoff be $g(z(t_1)) = \max(|z_1(t_1)|, |z_2(t_1)|)$. Calculation using the scheme expounded above show that the value of the game is of the following form:

$$G(z) = G_1(z) = \max(|z_1 + t_1 z_3|, |z_2 + t_1 z_4|) + t_1 - t_1^2 / 2 \quad \text{for } t_1 \leqslant 1$$
$$G(z) = \max(1/2, G_1(z)) \quad \text{for } t_1 > 1$$

Note that the successive procedures of constructing the function of the payoff value or of the minimax in the game of convergence at a specified time were considered, e.g., in /4,5/.

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